Distortion in Feedback Amplifiers

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Distortion in feedback amplifiers and other non-linear circuits is analyzed for the case where the magnitude and phase of the feedback varies with frequency. The analysis is limited to cases where the distortion products are small compared to the fundamentals and where the non-linear element can be described by a power series having only a few terms. However, many practical amplifiers are adequately described by the analysis. Formulae are derived for a number of third-order products and their dependence upon various feedbacks at second order frequencies is demonstrated.

INTRODUCTION

Distortion in feedback amplifiers has previously been studied for the case where the feedback is independent of frequency.[1] However, in many practical cases, the variation of feedback with frequency produces significant deviations from this simple theory.[2] The present analysis takes into account the magnitude and phase of the feedback at all frequencies in determining the amount of any particular modulation product. This analysis has proved useful in the design of amplifiers for the L3 coaxial carrier system[3, 4] as well as in the analysis of a number of non-linear circuits. The method is most useful in cases where the distortion products are small compared to the fundamental signals and where the non-linear element can be described by a power series having only a few terms. More complex cases can be treated but the labor involved is appreciably greater. However, many practical feedback amplifiers are adequately described by the analysis and, in addition, some understanding is obtained as to the mechanisms involved. In particular, the dependence of third order distortions on the feedbacks at second order frequencies is demonstrated and formulae are obtained.

THE PROBLEM

When a signal is sent through a non-linear element, such as a vacuum tube, the output can usually be described as a power series of the input signal. This is especially true in wideband amplifiers where plate load impedances are low and plate current is determined largely by grid-cathode voltage. Often the non-linear element is contained within a feedback loop such that a portion of the output is returned to the input. In this situation the total input contains a power series of the original input and the situation is considerably complicated. It is well-known,[1, 2] for example, that third harmonics can be produced not only by the cube term of the power series but, with feedback, by the square term. The square term produces second harmonic output which, after being fed back, mixes again with the fundamental to form (via the square again) third harmonics. Thus, the third harmonic output becomes dependent to some degree on the feedback at the second harmonic. This relationship becomes somewhat more complex when several fundamental inputs are present simultaneously but, in general, the output of a particular third order product depends on the feedback at, at least, some of the second order product frequencies. Within the limitations of the simplifying assumptions used, the present analysis evaluates these relationships.

THE METHOD AND THE ASSUMPTIONS

It is assumed that the non-linear element can be described by a power series of the form

$$i = a_1e + a_2e^2 + a_3e^3 + a_4e^4 + \cdots . \tag{1}$$

In a circuit such as shown on Fig. 1, e represents incremental grid-cathode voltage and i, incremental plate current of a vacuum tube. If, as shown on Fig. 1, a fraction of the output is returned to the input, then the grid-

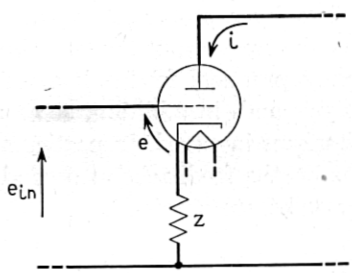

Fig. 1 — Feedback Amplifier Equivalent Circuit.

cathode voltage becomes the difference between the applied and fed back signals or,

$$e = e_{\text{in}} - iZ \tag{2}$$

While Z has the dimension of impedance, it is not limited to the ± 90 degrees of an ordinary two terminal impedance. In practice, $a_1 Z$ usually represents the loop gain, $(\mu\beta)$, of the feedback amplifier. Thus, Fig. 1 is used to represent the voltage and current relationships of any feedback amplifier.

Having expressed the output current as a power series of the grid-cathode voltage (1) and having expressed the grid cathode voltage as the sum of the input and fedback voltages, (2), the next step is to combine these expressions. The method consists of first expressing the input voltage, e_{in}, in terms of its cosinusoidal components. The grid-cathode voltage is expressed as a Fourier series of cosines having complex coefficients. The frequencies in the series represent all combinations and harmonics of the frequencies present in the input signal itself. The coefficients of the power series for the current, a_1, a_2, a_3, etc., are known from the tube characteristics. The problem concerns itself initially with finding the unknown coefficients for the Fourier series representation of the grid-cathode voltage.

The method is to first insert the Fourier series for e in the power series to find a Fourier series for i in terms of the coefficients for e. Then, equating the two sides of the equation, frequency by frequency, one obtains a set of simultaneous equations which can be solved for the coefficients of e and in turn for the coefficients of i. It so happens that with the assumptions used here the equations can be solved individually.

The chief difficulty of the method resides in the fact that both the power series and Fourier series are infinite and therefore a rigorous solution is impractical. Fortunately, simplifying assumptions permit limiting the series in many practical cases without seriously degrading the accuracy. The assumptions used here are as follows:

1. All distortion products are small compared to the fundamentals.
2. Third order distortion products are small compared to second order products.
3. The device non-linearity is adequately described by a simple three term power series. Fourth order and higher powers are neglected.
4. Frequency components representing fourth order and higher interactions are neglected.

"SINGLE-FREQUENCY INPUT"

In the single-frequency case the input signal, e_{in}, is given simply by

$$e_{\text{in}} = A \cos \alpha t \tag{3}$$

Since the input is a single frequency, the Fourier series representation of the grid-cathode voltage, e, is known to contain merely the harmonics of the input frequency. Thus,

$$e = \sum_{n=0}^{n=\infty} k_n \cos n\alpha t \tag{4}$$

or

$$e = k_0 + k_1 \cos \alpha t + k_2 \cos 2\alpha t + k_3 \cos 3\alpha t \cdots \tag{5}$$

The next step is to insert (5) in (1) limiting (1) to the first three terms as follows:

$$i = a_1 e + a_2 e^2 + a_3 e^3 \tag{6}$$

In doing this it must be remembered that k_n is complex and represents a complex voltage. Therefore, the products are not the ordinary result of the product of two complex numbers where the magnitudes multiply and and the angles add. Rather, these products, representing complex numbers pertaining to impedance, voltages or currents of different, or the same, frequencies are formed by using the conjugate of any coefficient whose frequency subtracts in the formation of the product frequency. For example,

$$\begin{aligned} k_a k_b \cos A \cos B &= \tfrac{1}{2} k_a k_b \cos (A + B) + \tfrac{1}{2} k_a \bar{k}_b \cos (A - B) \\ (\bar{\ }) &= \text{conjugate}, \qquad A > B \end{aligned} \tag{7}$$

If this rule were not followed at least the phases of the products would be incorrect.

Performing the operation for e^2 and combining terms of the same frequency yields,

$$\begin{aligned} e^2 = k_0^2 &+ \frac{k_1\bar{k}_1 + k_2\bar{k}_2 + k_3\bar{k}_3}{2} + (+2k_0k_1 + \bar{k}_1k_2 + \bar{k}_2k_3) \cos \alpha t \\ &+ (2k_0k_2 + \tfrac{1}{2}k_1^2 + k_3\bar{k}_1) \cos 2\alpha t + (2k_0k_3 + k_1k_2) \cos 3\alpha t \\ &+ (k_1k_3 + \tfrac{1}{2}k_2^2) \cos 4\alpha t + k_2k_3 \cos 5\alpha t + \tfrac{1}{2}k_3^2 \cos 6\alpha t \end{aligned} \tag{8}$$

A similar operation for e^3 yields

$$
\begin{aligned}
e^3 = {} & k_0^3 + \tfrac{1}{2}[k_0k_1\bar{k}_1 + k_0k_2\bar{k}_2 + k_0k_3\bar{k}_3] + \tfrac{1}{2}Re[2k_0(k_1^2 + k_2^2 + k_3^2) \\
& + k_1k_2\left(\bar{k}_1 + \frac{k_1}{2}\right) + \tfrac{1}{2}k_3(k_1k_2 + k_1\bar{k}_2 + \bar{k}_1k_2)] \\
& + ((+3k_0^2k_1 + 3k_0\bar{k}_1k_2 + 3k_0\bar{k}_2k_3 + \tfrac{3}{4}k_1^2\bar{k}_1 + \tfrac{3}{4}\bar{k}_1^2k_3 + \tfrac{3}{2}k_1k_2\bar{k}_2 \\
& + \tfrac{3}{2}k_1k_3\bar{k}_3 + \tfrac{3}{4}k_2^2\bar{k}_3)) \cos \alpha t \\
& + ((+3k_0^2k_2 + \tfrac{3}{2}k_0k_1^2 + 3k_0\bar{k}_1k_3 + \tfrac{3}{2}k_1\bar{k}_1k_2 + \tfrac{3}{2}k_1\bar{k}_2k_3 \\
& + \tfrac{3}{4}k_2^2\bar{k}_2 + \tfrac{3}{2}k_2k_3\bar{k}_3)) \cos 2\alpha t \qquad (9) \\
& + ((3k_0^2k_3 + 3k_0k_1k_2 + \tfrac{1}{4}k_1^3 + \tfrac{3}{2}k_1\bar{k}_1k_3 + \tfrac{3}{4}\bar{k}_1k_2^2 \\
& + \tfrac{3}{2}k_2\bar{k}_2k_3 + \tfrac{3}{4}\bar{k}_3k_3^2)) \cos 3\alpha t \\
& + ((\tfrac{3}{2}k_0k_2^2 + 3k_0k_1k_3 + \tfrac{3}{4}k_1^2k_2 + \tfrac{3}{2}\bar{k}_1k_2k_3 + \tfrac{3}{4}\bar{k}_2k_3^2)) \cos 4\alpha t \\
& + ((3k_0k_2k_3 + \tfrac{3}{4}k_1^2k_3 + \tfrac{3}{4}k_1k_2^2 + \tfrac{3}{4}\bar{k}_1k_3^2)) \cos 5\alpha t \\
& + ((\tfrac{3}{2}k_0k_3^2 + \tfrac{3}{2}k_1k_2k_3 + \tfrac{1}{4}k_2^3)) \cos 6\alpha t \\
& + (\tfrac{3}{4}k_1k_3^2 + \tfrac{3}{4}k_2^2k_3) \cos 7\alpha t + \tfrac{3}{4}k_2k_3^2 \cos 8\alpha t + \tfrac{1}{4}k_3^3 \cos 9\alpha t
\end{aligned}
$$

We now introduce the assumptions. Specifically,

$$k_1 \gg k_2 \gg k_3, \qquad k_1 \gg k_0 \gg k_3 \qquad (10)$$

and we neglect k_4, k_5, etc. The reduction in labor is apparent from inspection of equations 8 and 9. This simplifies equations (5), (8) and (9) as follows:

$$e = k_0 + k_1 \cos \alpha t + k_2 \cos 2\alpha t + k_3 \cos 3\alpha t \qquad (11)$$

$$e^2 = \tfrac{1}{2}k_1\bar{k}_1 + (2k_0k_1 + \bar{k}_1k_2) \cos \alpha t + \tfrac{1}{2}k_1^2 \cos 2\alpha t + k_1k_2 \cos 3\alpha t \qquad (12)$$

$$
\begin{aligned}
e^3 = {} & \tfrac{1}{2}k_0\bar{k}_1k_1 + Re\left[k_0k_1^2 + \tfrac{1}{2}k_1k_2\left(k_1 + \frac{\bar{k}_1}{2}\right)\right] + \tfrac{3}{4}k_1^2\bar{k}_1 \cos \alpha t \\
& + (\tfrac{3}{2}k_0k_1^2 + \tfrac{3}{2}k_1\bar{k}_1k_2) \cos 2\alpha t + \tfrac{1}{4}k_1^3 \cos 3\alpha t
\end{aligned} \qquad (13)
$$

From (2) and (6) we know

$$e = A \cos \alpha t - Z[a_1e + a_2e^2 + a_3e^3] \qquad (14)$$

which may be written as

$$A \cos \alpha t = (1 + Za_1)e + Za_2e^2 + Za_3e^3 \qquad (15)$$

Using the values for e, e^2, e^3, given by (11), (12) and (13), in (15) and solving, frequency by frequency, yields for dc

$$0 = (1 + Z_0a_1)k_0 + Z_0a_2\tfrac{1}{2}k_1\bar{k}_1 + Z_0a_3\left(\tfrac{1}{2}k_0\bar{k}_1k_1 + Re\left[k_0k_1^2 + \tfrac{1}{2}k_1k_2\left(k_1 + \frac{\bar{k}_1}{2}\right)\right]\right) \tag{16}$$

For α

$$A = (1 + Z_1a_1)k_1 + Z_1a_2(2k_0k_1 + \bar{k}_1k_2) + \tfrac{3}{4}Z_1a_3k_1^2\bar{k}_1 \tag{17}$$

For 2α

$$0 = (1 + Z_2a_1)k_2 + \tfrac{1}{2}Z_2a_2k_1^2 + Z_2a_3(\tfrac{3}{2}k_0k_1^2 + \tfrac{3}{2}k_1\bar{k}_1k_2) \tag{18}$$

For 3α

$$0 = (1 + Z_3a_1)k_3 + Z_3a_2k_1k_2 + \tfrac{1}{4}Z_3a_3k_1^3 \tag{19}$$

In order for assumptions 1 and 2 to be met, namely,

$$k_1 \gg k_0\,, \qquad k_2 \gg k_3\,, \qquad k_1 \gg k_2\,, \qquad k_0 \gg k_3\,, \tag{20}$$

it is necessary that

$$a_1e \gg a_2e^2 \gg a_3e^3 \tag{21}$$

or

$$a_1 \gg a_2e \gg a_3e^2. \tag{22}$$

Therefore, since

$$a_2k_1^2 \gg a_3k_1^2k_0 \qquad \text{or} \qquad a_3k_1^2k_2 \tag{23}$$

Equation 18 can be solved directly for k_2 yielding

$$k_2 = -\frac{1}{2}\frac{a_2Z_2}{1 + a_1Z_2}k_1^2 \tag{24}$$

and likewise

$$k_0 = -\frac{a_2Z_0}{1 + a_1Z_0}k_1\bar{k}_1 \tag{25}$$

This procedure avoids a simultaneous solution of (16) to (19). Since, for small distortion,

$$k_1 \approx \frac{A}{1 + a_1Z_1} \tag{26}$$

and having k_0 and k_2, one obtains

$$k_0 = -\frac{a_2}{2}\left(\frac{Z_0}{1 + a_1Z_0}\right)\left(\frac{A}{1 + a_1Z_1}\right)\overline{\left(\frac{A}{1 + a_1Z_1}\right)} \tag{27}$$

$$k_1 = \frac{A}{1 + a_1Z_1} - Z_1\left[\frac{3a_3}{4} - \frac{a_2^2Z_0}{1 + a_1Z_0} - \frac{a_2^2Z_2}{2(1 + a_1Z_2)}\right] \cdot \left(\frac{A}{1 + a_1Z_1}\right)^2\overline{\left(\frac{A}{1 + a_1Z_1}\right)} \tag{28}$$

$$k_2 = \frac{a_2}{2}\left(\frac{Z_2}{1 + a_1Z_2}\right)\left(\frac{A}{1 + a_1Z_1}\right)^2 \tag{29}$$

$$k_3 = \frac{Z_3}{1 + a_1Z_3}\left[\frac{a_3}{4} - \frac{a_2^2Z_2}{2(1 + a_1Z_2)}\right]\left(\frac{A}{1 + a_1Z_1}\right)^3 \tag{30}$$

The corresponding values for the output currents are readily obtained as,

$$i_0 = \frac{a_2}{2(1 + a_1Z_0)}\left(\frac{A}{1 + a_1Z_1}\right)\overline{\left(\frac{A}{1 + a_1Z_1}\right)} \tag{31}$$

$$i_1 = \left[\frac{a_1A}{1 + a_1Z_1} + \left(\frac{3a_3}{4} - \frac{a_2^2Z_0}{1 + a_1Z_0} - \frac{a_2^2Z_2}{2(1 + a_1Z_2)}\right) \cdot \left(\frac{A}{1 + a_1Z_1}\right)^2\overline{\left(\frac{A}{1 + a_1Z_1}\right)}\right]\cos \alpha t \tag{32}$$

$$i_2 = \frac{a_2}{2(1 + a_1Z_2)}\left(\frac{A}{1 + a_1Z_1}\right)^2 \cos 2\alpha t \tag{33}$$

$$i_3 = \frac{1}{1 + a_1Z_3}\left[\frac{a_3}{4} - \frac{a_2^2Z_2}{2(1 + a_1Z_2)}\right]\left(\frac{A}{1 + a_1Z_1}\right)^3 \cos 3\alpha t \tag{34}$$

The expression for the fundamental output current includes the third order distortion of fundamental frequency. This is often viewed as a gain change and, in order to keep the polarity positive, will be expressed as expansion or increase in gain. The expansion is defined here as the ratio of the gain to the gain at small signal levels. The gain at small signal levels is obviously found by

$$A \to 0, \qquad i_1 \to \frac{a_1A}{1 + a_1Z_1}\cos \alpha t \tag{35}$$

and therefore,

$$\text{Expansion} = \frac{i_1}{\dfrac{a_1A}{1 + a_1Z_1}\cos \alpha t} \tag{36}$$

Thus,

$$\text{Expansion} = 1 + \left[\frac{3a_3}{4a_1} - \frac{a_2^{\,2}Z_0}{a_1(1 + a_1Z_0)} - \frac{a_2^{\,2}Z_2}{2a_1(1 + a_1Z_2)}\right] \cdot \left(\frac{A}{1 + a_1Z_1}\right)\left(\overline{\frac{A}{1 + a_1Z_1}}\right) \quad (37)$$

It should be pointed out that both this solution and the others to follow can be applied to other non-linear circuits besides feedback amplifiers so long as the assumptions used are adequately satisfied in practice.[5] Fig. 2 shows a simple non-linear circuit consisting of the series combination of a generator, impedance, Z, and a non-linear element. If the non-linear element can be adequately described by the power series of (6), then the solution is the same as given above. Equation 2 is obviously the same and therefore, so long as the assumptions of (10), etc., are satisfied, (31), (32), (33) and (34) represent accurate expressions of the currents.

BALANCED PUSH-PULL AMPLIFIER

A balanced push-pull amplifier is not often thought of as a feedback structure. However, Fig. 3 shows such a circuit having a cathode feedback impedance Z, common to both sides. The cathode impedance is usually used for bias and sometimes to assist in balancing. This circuit has been analyzed to determine the effect of second order distortions which are fed back via the cathode impedance, Z, even though they do not appear in the load. Note that for the perfectly balanced case odd order components cancel and even orders add across the cathode impedance. Thus, one might think of this as a structure with feedback only at even order distortion components.

Proceeding in the same fashion as for the previous example and assuming perfect balance we write the power series for the plate currents

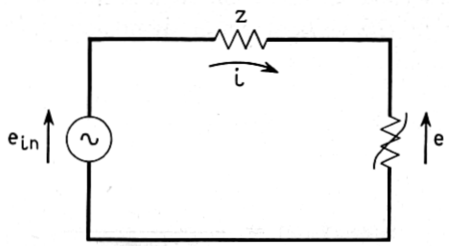

Fig. 2 — Non-linear circuit.

in terms of the grid-cathode voltages as

$$i_a = a_1 e_a + a_2 e_a^2 + a_3 e_a^3 \tag{38a}$$

$$i_b = a_1 e_b + a_2 e_b^2 + a_3 e_b^3 \tag{38b}$$

The input loops have the voltage relationships

$$e_a = A \cos \alpha t - Z(i_a + i_b) \tag{39a}$$

$$e_b = -A \cos \alpha t - Z(i_a + i_b) \tag{39b}$$

Finally, the grid-cathode voltages are expressed in Fourier series form.

$$e_a = \Sigma k_n \cos n\alpha t \tag{40a}$$

$$e_b = \Sigma j_n \cos n\alpha t \tag{40b}$$

Noting that the desired output is given by

$$\text{output current} = i_a - i_b \tag{41}$$

and observing that

$$k_1 \approx A \tag{42a}$$

$$j_1 \approx -A \tag{42b}$$

it can be shown that the dc output is

$$i_a|_{\text{dc}} - i_b|_{\text{dc}} = 0, \tag{43}$$

the fundamental is

$$i_a|_\alpha - i_b|_\alpha = \left(2a_1 A + 2A^2 \bar{A} \left[\frac{3a_3}{4} - \frac{2a_2^2 Z_0}{1 + 2a_1 Z_0} - \frac{a_2^2 Z_2}{1 + 2a_1 Z_2}\right]\right) \cos \alpha t \tag{44}$$

the second harmonic output current is

$$i_a|_{2\alpha} - i_b|_{2\alpha} = 0 \tag{45}$$

and the third harmonic output current is

$$i_a|_{3\alpha} - i_b|_{3\alpha} = A^3 \left[\frac{a_3}{2} - \frac{2a_2^2 Z_2}{1 + 2a_1 Z_2}\right] \cos 3\alpha t \tag{46}$$

The pattern here is the same as in the previous example, seconds are fed back to form thirds. Thus, while the balance removes second order products from the output, nevertheless, the level of third order distortion can be materially increased. Even if the cathode impedance, Z, is bypassed, the dc component can produce gain changes. This is equally true of the ordinary single-sided cathode-biased amplifier, another "non-feedback" amplifier.

APPLICATIONS OF THE THEORY

During the development of the L3 coaxial system some work was done on a non-linear circuit to generate modulation products. The experimental results failed to check with existing theory by large factors and the theory described here was developed to explain the difference.

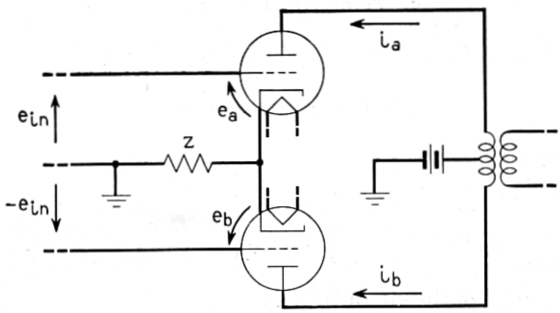

Fig. 3 — Balanced push-pull amplifier.

The deviations were then easily understood as, for example, a lack of feedback of direct current, a second-order product, yielding an expansion to third harmonic ratio appreciably different from three to one. With multiple frequency inputs the relative levels of other third order products were similarly affected. Application of the new theory which takes into account the magnitude and phase of the feedback yielded entirely satisfactory agreement.

The theory was later applied to various modulation characteristics of the L3 coaxial-system line amplifier.[3] In one case an effect was predicted by the theory which had not, at that time, been observed experimentally. In the L3 line amplifier the vacuum tubes are operated with local dc feedback from large cathode bias resistors suitably bypassed. The normal feedback loop does not extend to dc. The theory predicted that the compression would be reduced if this cathode dc feedback were removed. Typical values for the power series coefficients of the output stage (437A tubes) are

$$a_1 = 0.045$$

$$a_2 = 0.014$$

$$a_3 = -0.0057$$

Using equation 37 and the approximation of large second order feedback, ($a_1Z_0 \gg 1$, $a_1Z_2 \gg 1$), the expansion is found to be

$$\text{expansion} = 1 - 0.240 \left(\frac{A}{1 + a_1Z_1}\right)\left(\overline{\frac{A}{1 + a_1Z_1}}\right)$$

When the dc feedback is removed leaving the second harmonic value unchanged, one calculates

$$\text{expansion} = 1 - 0.1433 \left(\frac{A}{1 + a_1Z_1}\right)\left(\overline{\frac{A}{1 + a_1Z_1}}\right)$$

The ratio of the calculated compression (negative expansion) increment is

$$\frac{0.240}{0.1433} = 1.673 = 4.4 \text{ db calculated}$$

Thus the calculation indicates that removal of the dc feedback would reduce the compression produced by the amplifier by 4.4 db.

The incremental loss in gain of the amplifier was measured with both normal operation and with the local dc feedback removed. The removal of the dc feedback reduced the measured compression increment by the ratio.

$$\frac{0.0945}{0.058} = 1.63 = 4.2 \text{ db measured}$$

Thus the predicted effect was verified and the amount confirmed. It might be noted that the dc feedback was retained in the design in spite of the somewhat greater compression because of its great value in stabilizing the current and transconductance of the tubes against aging effects.

A third application of the theory has been in the design of the ambient temperature compensation oscillator used in the pilot regulators of the L3 coaxial system.[4] Here it was necessary to obtain compression in an amplifier whose tube had a positive a_3 and tended to expand. Gain could not be expended on signal frequency feedback but, by the use of 20 db of dc feedback, compression from second order dc feedback was made greater than the expansion effect. Thus the amplifier compressed (lost gain) as the signal increased. This effect, while small, nevertheless solved a serious motorboating problem. It also points up a technique by which gain changes with level can be balanced out. If a tube having a positive a_3 is provided with an appropriate amount of second order feedback (which always produces compression), then the two effects can be cancelled. This yields a gain which, to a good approximation, is independent of signal level. This is quite different from the technique of using a 90° feedback to convert "μ" gain changes into phase changes. Here there need be no gain or phase changes produced and feedback at the signal frequencies is unnecessary. However, the balance must be adjusted for the particular tube's modulation coefficients.

CONCLUSIONS

The analysis of cases with more than one input frequency are treated in the Appendix. Formulas are derived for second and third order products for up to three input frequencies. Certain of these results have also been expressed in terms of modulated tones to determine gain changes of sidebands relative to the carrier, etc. Comments on these results are included below.

Examining (31) to (34) pertaining to the output current for a single input frequency, it is interesting to note that, within the approximations, second order output is not affected by the feedback at other product frequencies. Of course, if the second order distortion, (31 and 33), were not large compared to the thirds this would no longer be the case. Third order outputs, (32), (34), show that the relative contributions of various seconds in forming thirds is not equal. For expansion the dc effect is normally twice the second harmonic effect while for third harmonic only the second harmonic contributes. Thus, the ratio of third harmonic to expansion, normally thought of as ⅓, can vary widely depending on the relative feedbacks at dc and the second harmonic.

In general, it can be stated that the level of a given third order product is not an accurate indication of other third order products nor is it always a good indication of the same product at another set of frequencies.

In carrier systems products such as $2\alpha - \beta$ and $\alpha + \beta - \gamma$ tend to add by voltage when the fundamentals (and their products) are closely spaced because insufficient phase distortion occurs to break up the in-phase addition among amplifiers. Thus, products of this type often tend to dominate the linearity problem. These products involve frequency differences $\alpha - \beta$ and $\beta - \gamma$ for example, which fall at low frequencies where the second order feedback effects should be subject to control. Thus, by appropriate use of phase shifts or small feedback or even small positive feedback at these low (outside the band) frequencies it may, in some cases, be possible to either reduce the level of these products or break up their tendency to add by voltage.

In line with the above comment it should be noted that any tendency towards instability of the feedback loop can lead to abnormally high distortion levels. A second order interaction frequency may suffer from a $Z/(1 + a_1Z)$ factor in excess of $1/a_1$. Typically Z has an angle of 150 degrees or so in the cutoff region of a feedback loop. For large Z the factor is $1/a_1$ and independent of Z. However, where a_1Z drops to, say, $2/\sqrt{3}$ the magnitude of $Z/(1 + a_1Z)$ becomes $2/a_1$ at an angle of 60 degrees. Whether this doubles the interaction effect depends, of course, on the phase of the other terms but, in many cases, the effect will be appreciable. This example was for a feedback loop of conservative design. Where $1 + a_1Z$ approaches zero more closely the effect will be larger. Thus, an amplifier having poor stability margins may exhibit unusual modulation behavior for third order products involving second order differences falling in the frequency range of the poor stability margin.

Even in cases where the amplifier is not considered as having feedback, second order products may be returned to the input at frequencies outside the useful range to affect in-band products. This effect is perhaps most obvious in the dc case where expansion or compression is most likely to be affected. Such a dc feedback can, for example, cause the amplification of a short pulse to differ from that of a steady tone. Alternatively, a suddenly applied tone may, at first, produce one output level and then, following a transient dependent upon the cutoff of the dc feedback, settle down at a different level. Such effects are difficult to predict rigorously since they involve essentially a large number of input frequencies. However, the mechanisms involved are easily understood.

In the case of the balanced push-pull amplifier and similar circuits, second order distortion products can be fed back via stray paths without feedback of fundamentals. This can produce significant increases in third order distortion compared to a single-sided amplifier. This is analogous to the well-known fact that putting feedback around an amplifier rarely reduces third order products by the amount of the feedback. The reason is the same, the circuit change allows seconds to feed back and make thirds.

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BIBLIOGRAPHY

1. Llewellyn, F. B., Operation of Thermionic Vacuum Tube Circuits, B.S.T.J., **5**, pp. 433–462, July, 1926.
2. Llewellyn, F. B., Constant Frequency Oscillators, B.S.T.J., **11**, pp. 67–100, Jan., 1932.
3. Morris, L. H., Lovell, G. H., and Dickinson, F. R., The L3 Coaxial System — Amplifiers, B.S.T.J., **32**, pp. 879–914, July 1953.
4. Ketchledge, R. W. and Finch, T. R., The L3 Coaxial System — Equalization and Regulation, B.S.T.J., **32**, pp. 833–878, July 1953.
5. Gillies, A. W., The Application of Power Series to the Solution of Non-Linear Circuit Problems, Proc. I.E.E., **96**, No. 44, Part III, Nov., 1949.

Appendix

THREE FREQUENCY INPUT

In the case of three input frequencies the derivation of particular products is somewhat more complicated than in the case of a single frequency input and the number of products of interest tends to be greater. However, the methods and the assumptions are essentially the same.

The input signal is assumed to be of the form

$$e_{\text{in}} = A \cos \alpha t + B \cos \beta t + C \cos \gamma t \qquad \alpha > \beta > \gamma \tag{A1}$$

and the power series for the tube is still

$$i = a_1 e + a_2 e^2 + a_3 e^3 \tag{A2}$$

The loop equation (Fig. 1) is unchanged,

$$e = e_{\text{in}} - iZ \tag{A3}$$

The Fourier series for the grid-cathode voltage is taken as

$$e = \sum_{n\,pq} k_{n,p,q} \cos (n\alpha + p\beta + q\gamma), \qquad \alpha > \beta > \gamma \tag{A4}$$

which may be written as

$$\begin{aligned} e = {} & k_{0,0,0} + k_{1,0,0} \cos \alpha t + k_{0,1,0} \cos \beta t + k_{0,0,1} \cos \gamma t \\ & + k_{2,0,0} \cos 2\alpha t + k_{0,2,0} \cos 2\beta t + k_{0,0,2} \cos 2\gamma t \\ & + k_{1,-1,0} \cos (\alpha - \beta)t + k_{1,0,-1} \cos (\alpha - \gamma)t + k_{0,1,-1} \cos (\beta - \gamma)t \\ & + k_{1,1,0} \cos (\alpha + \beta)t + k_{1,0,1} \cos (\alpha + \gamma)t + k_{0,1,1} \cos (\beta + \gamma)t \\ & + k_{3,0,0} \cos 3\alpha t + k_{0,3,0} \cos 3\beta t + k_{0,0,3} \cos 3\gamma t \\ & + k_{1,1,1} \cos (\alpha + \beta + \gamma)t + k_{1,1,-1} \cos (\alpha + \beta - \gamma)t + \text{etc., etc., etc.} \end{aligned} \tag{A5}$$

In the above Fourier series for e, the dc through third order products total 32 terms. Consequently, the formation of e^2 involves 32^2 or approximately 1,000 multiplications. To form e^3 requires 32^3 or approximately

30,000 multiplications. Since the labor involved is obviously excessive the technique used is to select only the dominant terms in forming the desired products. In particular, the assumption that fundamentals are large compared to second order products and that these, in turn, are large compared to third order products is used repeatedly.

A typical third order product for a three frequency input is the one having the frequency $\alpha + \beta - \gamma$. In order to find the amplitude of this product the dominant terms in both e^2 and e^3 are selected by inspection. In e^2 the frequency, $\alpha + \beta - \gamma$, can be formed by a large variety of combinations. Some of these are

γ	and	$\alpha + \beta$	$\alpha + \gamma$	and	$2\alpha + \beta$
$\alpha + \beta$	and	γ	dc	and	$\alpha + \beta - \gamma$
$\beta - \gamma$	and	α	$\alpha + \beta - \gamma$	and	dc
α	and	$\beta - \gamma$	$2\alpha - \gamma$	and	$\alpha - \beta$
β	and	$\alpha - \gamma$	$\alpha - \beta$	and	$2\alpha - \gamma$
$\alpha - \gamma$	and	β		etc.	
$2\alpha + \beta$	and	$\alpha + \gamma$			

Recognizing that the dominant products are fundamentals $\times$ seconds we can write the e^2 terms of frequency $\alpha + \beta - \gamma$ as follows:

$$e^2 \,|_{\alpha+\beta-\gamma} = 2k_{1,1,0}k_{0,0,1} \cos (\alpha + \beta)t \cos \gamma t + 2k_{0,1,-1}k_{1,0,0} \cos (\beta - \gamma)t \cos \alpha t + 2k_{1,0,-1}k_{0,1,0} \cos (\alpha - \gamma)t \cos \beta t \tag{A6}$$

When the indicated multiplications are carried out (A6) simplifies to the form given below. Note that the conjugate is used whenever its corresponding frequency subtracts in the formation of the desired product.

$$e^2 \,|_{\alpha+\beta-\gamma} = [k_{1,1,0}\overline{k_{0,0,1}} + k_{0,1,-1}k_{1,0,0} + k_{1,0,-1}k_{0,1,0}] \cos (\alpha + \beta - \gamma)t \tag{A7}$$

In like manner the e^3 terms of frequency $\alpha + \beta - \gamma$ are observed to be dominated by fundamentals $\times$ fundamentals $\times$ fundamentals. Namely,

α	β	γ
α	γ	β
β	α	γ
β	γ	α
γ	α	β
γ	β	α

Thus,

$$e^3 |_{\alpha+\beta-\gamma} = \tfrac{6}{4} k_{1,0,0} k_{0,1,0} \overline{k_{0,0,1}} \cos(\alpha + \beta - \gamma)t \tag{A8}$$

The coefficients for the fundamentals themselves are easily approximated in the same manner as was used to arrive at (26). However, it is still necessary to find the second order coefficients of e used in forming e^2 in (A7). These second order products are dominated by fundamentals x fundamentals and can therefore easily be shown to be

$$k_{1,1,0} = \frac{-a_2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}} k_{1,0,0} k_{0,1,0} \tag{A9}$$

$$k_{0,1,-1} = \frac{-a_2 Z_{\beta-\gamma}}{1 + a_1 Z_{\beta-\gamma}} k_{0,1,0} \overline{k_{0,0,1}} \tag{A10}$$

$$k_{1,0,-1} = \frac{-a_2 Z_{\alpha-\gamma}}{1 + a_1 Z_{\alpha-\gamma}} k_{1,0,0} \overline{k_{0,0,1}} \tag{A11}$$

Since there is no original input signal of frequency $\alpha + \beta - \gamma$, (A3) can be written as

$$e = -Z_{\alpha+\beta-\gamma}[a_1 e + a_2 e^2 + a_3 e^3]_{\alpha+\beta-\gamma} \tag{A12}$$

The subscript on both the impedance and the power series should be interpreted respectively as the value at this particular frequency and the content of this particular frequency. Inserting the appropriate values for e, e^2 and e^3 of frequency $\alpha + \beta - \gamma$ in (A12) one obtains,

$$\begin{aligned} k_{1,1,-1}(1 + a_1 Z_{\alpha+\beta-\gamma}) = -Z_{\alpha+\beta-\gamma} \Bigg[& \frac{-a_2^2}{1 + a_1 Z_{\alpha+\beta}} k_{1,0,0} k_{0,1,0} \overline{k_{0,0,1}} \\ & + \frac{-a_2^2}{1 + a_1 Z_{\beta-\gamma}} k_{0,1,0} \overline{k_{0,0,1}} k_{1,0,0} + \frac{-a_2^2}{1 + a_1 Z_{\alpha-\gamma}} k_{1,0,0} \overline{k_{0,0,1}} k_{0,1,0} \\ & + a_3 \frac{6}{4} k_{1,0,0} k_{0,1,0} \overline{k_{0,0,1}} \Bigg] \end{aligned} \tag{A13}$$

Note that

$$i_{\alpha+\beta-\gamma} = \frac{-k_{1,1,-1}}{Z_{\alpha+\beta-\gamma}} \tag{A14}$$

Thus,

$$\begin{aligned} i_{\alpha+\beta-\gamma} = \frac{1}{1 + a_1 Z_{\alpha+\beta-\gamma}} \Bigg[& \frac{3a_3}{2} - \frac{a_2^2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}} - \frac{a_2^2 Z_{\beta-\gamma}}{1 + a_1 Z_{\beta-\gamma}} \\ & - \frac{a_2^2 Z_{\alpha-\gamma}}{1 + a_1 Z_{\alpha-\gamma}} \Bigg] k_{1,0,0} \overline{k_{0,0,1}} k_{0,1,0} \cos(\alpha + \beta - \gamma)t \end{aligned} \tag{A15}$$

Approximating the fundamentals by

$$k_{1,0,0} = \frac{A}{1 + a_1 Z_\alpha} \tag{A16}$$

$$k_{0,1,0} = \frac{B}{1 + a_1 Z_\beta} \tag{A17}$$

$$k_{0,0,1} = \frac{C}{1 + a_1 Z_\gamma} \tag{A18}$$

We obtain for the desired product

$$\begin{aligned} i_{\alpha+\beta-\gamma} = {} & \frac{1}{1 + a_1 Z_{\alpha+\beta-\gamma}} \left[\frac{3a_3}{2} - \frac{a_2^2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}} - \frac{a_2^2 Z_{\beta-\gamma}}{1 + a_1 Z_{\beta-\gamma}} \right. \\ & \left. - \frac{a_2^2 Z_{\alpha-\gamma}}{1 + a_1 Z_{\alpha-\gamma}} \right] \left(\frac{A}{1 + a_1 Z_\alpha} \right) \left(\frac{B}{1 + a_1 Z_\beta} \right) \left(\overline{\frac{C}{1 + a_1 Z_\gamma}} \right) \\ & \cos (\alpha + \beta - \gamma) t \end{aligned} \tag{A19}$$

A similar derivation for the $\alpha + \beta + \gamma$ product yields the same expression except that the conjugate is removed and $+\gamma$ replaces $-\gamma$ on the impedance and frequency subscripts wherever $-\gamma$ appears above. Thus,

$$\begin{aligned} i_{\alpha+\beta+\gamma} = {} & \frac{1}{1 + a_1 Z_{\alpha+\beta+\gamma}} \left[\frac{3a_3}{2} - \frac{a_2^2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}} - \frac{a_2^2 Z_{\beta+\gamma}}{1 + a_1 Z_{\beta+\gamma}} \right. \\ & \left. - \frac{a_2^2 Z_{\alpha+\gamma}}{1 + a_1 Z_{\alpha+\gamma}} \right] \left(\frac{A}{1 + a_1 Z_\alpha} \right) \left(\frac{B}{1 + a_1 Z_\beta} \right) \left(\frac{C}{1 + a_1 Z_\gamma} \right) \\ & \cos (\alpha + \beta + \gamma) t \end{aligned} \tag{A20}$$

Another product of interest for three frequency inputs is expansion. As a typical case, the expansion of β in the presence of α, β, and γ is derived below. The method used relies on the same assumptions. Since the distortion product has the frequency β, in e^2, the following fundamentals x seconds terms dominate.

α	and	$\alpha - \beta$	dc	and	β
$\alpha - \beta$	and	α	γ	and	$\beta - \gamma$
$\alpha + \beta$	and	α	$\beta - \gamma$	and	γ
2β	and	β	γ	and	$\beta + \gamma$
β	and	2β	$\beta + \gamma$	and	γ
β	and	dc			

This simplifies to

$$e^2 |_\beta = [2k_{0,0,0}k_{0,1,0} + k_{0,2,0}\overline{k_{0,1,0}} + k_{1\,1,0}\overline{k_{1,0,0}} + k_{1,0,0}\overline{k_{1,-1,0}} + k_{0,1,-1}k_{0,0,1} + k_{0,1,1}\overline{k_{0,0,1}}] \cos \beta t \quad \text{(A21)}$$

In e^3 fundamentals × fundamentals × fundamentals dominate as follows,

α	β	α	3 terms
γ	β	γ	3 terms
β	β	β	1 term

This yields

$$e^3 |_\beta = \tfrac{3}{2}k_{1.0.0}\overline{k_{1,0,0}}k_{0,1,0} + \tfrac{3}{2}k_{0,0,1}\overline{k_{0,0,1}}k_{0,1,0} + \tfrac{3}{4}k_0{}^2{}_{,1,0}\overline{k_{0,1,0}} \quad \text{(A22)}$$

To proceed further again requires the determination of a number of second order products appearing in e^2. Note that these are the products which when fed back will beat with fundamentals to form the desired third order product. Since these second order products are dominated again by fundamentals × fundamentals they are easily shown to be

$$k_{0,0,0} = \frac{-a_2Z_0}{2(1 + a_1Z_0)} (k_{1,0,0}\overline{k_{1,0,0}} + k_{0,1,0}\overline{k_{0,1,0}} + k_{0,0\,1}\overline{k_{0,0,1}}) \quad \text{(A23)}$$

$$k_{0,2,0} = \frac{-a_2Z_{2\beta}}{2(1 + a_1Z_{2\beta})} (k_{0\,1,0})^2 \quad \text{(A24)}$$

$$k_{1,1,0} = \frac{-a_2Z_{\alpha+\beta}}{(1 + a_1Z_{\alpha+\beta})} k_{1,0,0}k_{0,1,0} \quad \text{(A25)}$$

$$k_{1,-1,0} = \frac{-a_2Z_{\alpha-\beta}}{1 + a_1Z_{\alpha-\beta}} k_{1,0,0}\overline{k_{0,1,0}} \quad \text{(A26)}$$

$$k_{0,1,-1} = \frac{-a_2Z_{\beta-\gamma}}{1 + a_1Z_{\beta-\gamma}} k_{0,1,0}\overline{k_{0,0,1}} \quad \text{(A27)}$$

$$k_{0,1,1} = \frac{-a_2Z_{\beta+\gamma}}{1 + a_1Z_{\beta+\gamma}} k_{0,1,0}k_{0,0,1} \quad \text{(A28)}$$

The next step is to rewrite (A3) as

$$k_{0,1,0} = B - Z_\beta(a_1k_{0,1,0} + a_2e^2 |_\beta + a_3e^3 |_\beta) \quad \text{(A29)}$$

which reduces to

$$k_{0,1,0} = \frac{B}{1 + a_1Z_\beta} - \frac{Z_\beta}{1 + a_1Z_\beta} (a_2e^2 |_\beta + a_3e^3 |_\beta) \quad \text{(A30)}$$

Substituting the terms of e^2 and e^3 of frequency β we obtain,

$$k_{0,1,0} = \frac{B}{1 + a_1 Z_\beta} - \frac{Z_\beta}{1 + a_1 Z_\beta}\Bigg[(a_2 k_{0,1,0})\Bigg(\frac{-a_2 Z_0}{1 + a_1 Z_0}(k_{1,0,0}\overline{k_{1,0,0}} + k_{0,1,0}\overline{k_{0,1,0}} + k_{0,0,1}\overline{k_{0,0,1}}) - \frac{a_2 Z_{2\beta}}{2(1 + a_1 Z_{2\beta})} k_{0,1,0}\overline{k_{0,1,0}}$$
$$- \frac{a_2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}} k_{1,0,0}\overline{k_{1,0,0}} - \frac{a_2 Z_{\alpha-\beta}}{1 + a_1 Z_{\alpha-\beta}} k_{1,0,0}\overline{k_{1,0,0}} \tag{A31}$$
$$- \frac{a_2 Z_{\beta-\gamma}}{1 - a_1 Z_{\beta-\gamma}} k_{0,0,1}\overline{k_{0,0,1}} - \frac{a_2 Z_{\beta+\gamma}}{1 + a_1 Z_{\beta+\gamma}} k_{0,0,1}\overline{k_{0,0,1}}\Bigg)$$
$$+ a_3 k_{0.1,0}(\tfrac{3}{4} k_{0,1\cdot 0}\overline{k_{0,1,0}} + \tfrac{3}{2} k_{1,0,0}\overline{k_{1,0,0}} + \tfrac{3}{2} k_{0,0,1}\overline{k_{0,0.1}})\Bigg]$$

Making use of values of the fundamentals as given in (16), (17), (18) and the relationship

$$i_\beta = -\frac{k_{0,1,0} - B}{Z_\beta} \tag{A32}$$

we find for the output current of frequency β

$$i_\beta = \frac{a_1 B}{1 + a_1 Z_\beta}\Bigg[1 + \frac{1}{1 + a_1 Z_\beta}\left(\frac{B}{1 + a_1 Z_\beta}\right)\left(\overline{\frac{B}{1 + a_1 Z_\beta}}\right)\Bigg(\frac{3a_3}{4a_1} - \frac{a_2^2 Z_0}{a_1(1 + a_1 Z_0)} - \frac{a_2^2 Z_{2\beta}}{2a_1(1 + a_1 Z_{2\beta})}\Bigg) + \frac{1}{1 + a_1 Z_\beta}\left(\frac{A}{1 + a_1 Z_\alpha}\right)$$
$$\left(\overline{\frac{A}{1 + a_1 Z_\alpha}}\right)\Bigg(\frac{3a_3}{2a_1} - \frac{a_2^2 Z_0}{a_1(1 + a_1 Z_0)} - \frac{a_2^2 Z_{\alpha+\beta}}{a_1(1 + a_1 Z_{\alpha+\beta})} \tag{A33}$$
$$- \frac{a_2^2 Z_{\alpha-\beta}}{a_1(1 + a_1 Z_{\alpha-\beta})}\Bigg) + \frac{1}{1 + a_1 Z_\beta}\left(\frac{C}{1 + a_1 Z_\gamma}\right)\left(\overline{\frac{C}{1 + a_1 Z_\gamma}}\right)$$
$$\Bigg(\frac{3a_3}{2a_1} - \frac{a_2^2 Z_0}{a_1(1 + a_1 Z_0)} - \frac{a_2^2 Z_{\beta-\gamma}}{a_1(1 + a_1 Z_{\beta-\gamma})} - \frac{a_2^2 Z_{\beta+\gamma}}{a_1(1 + a_1 Z_{\beta+\gamma})}\Bigg)\Bigg]\cos\beta t$$

The gain expansion is readily obtained from the above using the relationship

$$\text{gain expansion} = \frac{i_\beta}{\dfrac{a_1 B}{1 + a_1 Z_\beta}\cos\beta t} \tag{A34}$$

It is interesting to note that the influences of the other two fundamentals involve feedbacks at sum and difference frequencies relative to

β whereas the corresponding term for β involves the second harmonic and dc feedback. The latter also appears with the other fundamentals.

MODULATED CARRIER EXPANSION

The preceding analysis of three frequency inputs can also be applied to the situation where the signal consists of an ordinary amplitude modulated carrier. Such a modulated carrier consists of a carrier frequency with two sideband frequencies, one above and one below the carrier. To simplify the results given below it is assumed that the feedback is the same for the sidebands as it is for the carrier. The index of modulation is expressed as m and the modulating frequency is expressed as Δ. Thus, in the notation of the preceding section one has the relationships

$$A = C = \frac{m}{2} B \tag{A35}$$

$$\alpha = \beta + \Delta \tag{A36}$$

$$\gamma = \beta - \Delta \tag{A37}$$

$$Z_{\beta+\Delta} = Z_\beta = Z_{\beta-\Delta}$$

It is further assumed that the feedback is unchanged between the second harmonics of the sidebands and carrier so that

$$Z_{2(\beta+\Delta)} = Z_{2\beta} = Z_{2(\beta-\Delta)} \tag{A38}$$

Taking (A33) and equivalent expressions for i_α and i_γ it may be shown that

$$\begin{aligned}\text{gain expansion of the carrier} &= 1 \\ &+ \frac{1}{1 + a_1 Z_\beta}\left(\frac{B}{1 + a_1 Z_\beta}\right)\left(\overline{\frac{B}{1 + a_1 Z_\beta}}\right)\left[\frac{3a_3}{4a_1}(1 + m^2)\right. \\ &- \frac{a_2^2 Z_0}{a_1(1 + a_1 Z_0)}\left(1 + \frac{m^2}{2}\right) - \frac{a_2^2 Z_{2\beta}}{a_1(1 + a_1 Z_{2\beta})}\left(\frac{1 + m^2}{2}\right) \\ &\left. - \frac{a_2^2 Z_\Delta}{a_1(1 + a_1 Z_\Delta)}\left(\frac{m^2}{2}\right)\right]\end{aligned} \tag{A39}$$

$$\begin{aligned}\text{gain expansion of either sideband} &= 1 \\ &+ \frac{1}{1 + a_1 Z_\beta}\left(\frac{B}{1 + a_1 Z_\beta}\right)\left(\overline{\frac{B}{1 + a_1 Z_\beta}}\right)\left[\frac{3a_3}{2a_1}\left(1 + \frac{3}{8} m^2\right)\right. \\ &- \frac{a_2^2 Z_0}{a_1(1 + a_1 Z_0)}\left(1 + \frac{m^2}{2}\right) - \frac{a_2^2 Z_{2\beta}}{a_1(1 + a_1 Z_{2\beta})}\left(1 + \frac{3m^2}{8}\right) \\ &\left. - \frac{a_2^2 Z_\Delta}{a_1(1 + a_1 Z_\Delta)}(1) - \frac{a_2^2 Z_{2\Delta}}{a_1(1 + a_1 Z_{2\Delta})}\left(\frac{m^2}{4}\right)\right]\end{aligned} \tag{A40}$$

and the expansion of the modulation index is

$$\text{gain expansion of sideband relative to carrier} = 1 + \frac{1}{1 + a_1 Z_\beta}\left(\frac{B}{1 + a_1 Z_\beta}\right)\left(\overline{\frac{B}{1 + a_1 Z_\beta}}\right)\left[\frac{3a_3}{4a_1}\left(1 - \frac{m^2}{4}\right) - \frac{a_2^2 Z_{2\beta}}{a_1(1 + a_1 Z_{2\beta})}\left(\frac{1}{2} - \frac{m^2}{8}\right) - \frac{a_2^2 Z_{2\Delta}}{a_1(1 + a_1 Z_\Delta)}\left(1 - \frac{m^2}{2}\right) - \frac{a_2^2 Z_{2\Delta}}{a_1(1 + a_1 Z_{2\Delta})}\left(\frac{m^2}{4}\right)\right] \tag{A41}$$

Note that the dc term drops out of the expansion of the sidebands relative to the carrier because the dc term affects them equally.

TWO FREQUENCY INPUT

The third order products, $2\alpha \pm \beta$, are often of importance in carrier systems. They cannot be found simply by substitution of variables in (A19) or (A20) since, for example, α, β, γ, can be formed in six ways and α, α, β, in only three. Thus, the a_3 term has a coefficient of ¾ instead of 6/4. This is intended to sound a note of caution in merely changing variables to find other products.

Carrying through the complete calculations for $2\alpha \pm \beta$ one obtains,

$$i_{2\alpha-\beta} = \frac{1}{1 + a_1 Z_{2\alpha-\beta}}\left(\frac{A}{1 + a_1 Z_\alpha}\right)^2\left(\overline{\frac{B}{1 + a_1 Z_\beta}}\right)\left[\frac{3a_3}{4} - \frac{a_2^2 Z_{2\alpha}}{2(1 + a_1 Z_{2\alpha})} - \frac{a_2^2 Z_{\alpha-\beta}}{1 + a_1 Z_{\alpha-\beta}}\right]\cos(2\alpha - \beta)t \tag{A42}$$

and

$$i_{2\alpha+\beta} = \frac{1}{1 + a_1 Z_{2\alpha+\beta}}\left(\frac{A}{1 + a_1 Z_\alpha}\right)^2\left(\frac{B}{1 + a_1 Z_\beta}\right)\left[\frac{3a_3}{4} - \frac{1}{2}\frac{a_2^2 Z_{2\alpha}}{(1 + a_1 Z_{2\alpha})} - \frac{a_2^2 Z_{\alpha+\beta}}{1 + a_1 Z_{\alpha+\beta}}\right]\cos(2\alpha + \beta)t \tag{A43}$$